The DOZZ formula

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Outline

1. Gaussian multiplicative chaos
2. Liouville Conformal Field Theory (LCFT)
Plan of the talk

1. Gaussian multiplicative chaos
2. Liouville Conformal Field Theory (LCFT)
Gaussian multiplicative chaos (GMC)

Let $X$ be a centered log-correlated Gaussian field in a (bounded) domain of $\mathbb{R}^d$

$$\mathbb{E}[X(x)X(y)] = \ln \frac{1}{|x - y|} + g(x, y) \quad (g \text{ bounded})$$

**Theorem (Kahane, 1985)**

Let $X_\epsilon$ be a reasonable regularization of $X$.

1. **the limit**

   $$M_\gamma(dz) := e^{\gamma X(z)} dz := \lim_{\epsilon \to 0} \epsilon^{\gamma^2/2} e^{\gamma X_\epsilon(z)} dz$$

   exists in probability in the space of Radon measures.

2. **the limit does not depend on the regularization procedure.** $M_\gamma$ is different from zero if and only if $\gamma < \sqrt{2d}$.

See Berestycki’s recent article for a nice recent review and elementary approach (or Robert-Vargas, Saksman-Junnila, Shamov, R.-Vargas).
Intermittency properties

This type of measures appear in turbulence, finance, and field theory...

The parameter $\gamma$ rules the intermittency strength
What happens for critical $\gamma = \sqrt{2d}$?

Theorem (Duplantier, Rhodes, Sheffield, Vargas, 2012)

Let $X_\epsilon$ be a reasonable regularization of $X$. Then the limit

$$M_{\gamma=\sqrt{2d}}(dz) := e^{\sqrt{2d}X(z)} \, dz := \lim_{\epsilon \to 0} (\ln \epsilon)^{1/2} \epsilon^d e^{\sqrt{2d}X_\epsilon(z)} \, dz$$

exists in probability in the space of Radon measures and is non trivial.

**Question**: is it possible to get integrability results for these GMC measures?
Asymptotics of the maximum

**Question:** What is the law of the maximum of the maximum of the GFF?

**Theorem (Bramson/Ding/Zeitouni or Madaule 13’)**

Take a domain $D$

\[
\sup_{x \in D} X_\epsilon(x) - \sqrt{2d} \ln \frac{1}{\epsilon} + \frac{3}{2\sqrt{2d}} \ln \ln \frac{1}{\epsilon} \to Y, \quad \text{in law as } \epsilon \to 0.
\]

with

\[Y \overset{\text{law}}{=} \text{Gumbel} + C + \frac{1}{\sqrt{2d}} \ln M_{\gamma=\sqrt{2d}}(D).\]

We need here a description of the law of $\ln M_{\gamma=\sqrt{2d}}(D)$. 


Fyodorov-Bouchaud Formula

Let $X$ be a Gaussian field on the unit circle

$$
\mathbb{E}[X(\theta)X(\theta')] = \ln \left| \frac{1}{e^{i\theta} - e^{i\theta'}} \right|
$$

1. the measure $M_\gamma = e^{\gamma X(z)} \, dz$ admits moments of order $p < \frac{2}{\gamma^2}$

$$
\mathbb{E}[M_\gamma(0, 2\pi)^p] < +\infty \iff p < \frac{2}{\gamma^2}.
$$

2. integer moments $p \in \mathbb{N}$ (with $p < \frac{2}{\gamma^2}$) are easily computed
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$$\mathbb{E}[M_\gamma(0, 2\pi)^p] = \int_{[0, 2\pi]^p} \mathbb{E} \prod_{1 \leq k \leq p} e^{\gamma X(\theta_k)} - \frac{\gamma^2}{2}\mathbb{E}[X(\theta_k)^2] \, d\theta_k$$

$$= \int_{[0, 2\pi]^n} \prod_{1 \leq k < p \leq n} \frac{1}{|e^{i\theta_k} - e^{i\theta_p}|^{\gamma^2}} \, d\theta_1 \ldots d\theta_n$$

$$= \frac{\Gamma(1 - p\gamma^2/2)}{\Gamma(1 - \gamma^2/2)^p}$$
Fyodorov-Bouchaud Formula

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2. Integer moments $p \in \mathbb{N}$ (with $p < \frac{2}{\gamma^2}$) are easily computed

the Bouchaud Fyodorov proposal

$$\forall p < \frac{2}{\gamma^2}, \quad \mathbb{E}[M_\gamma(0, 2\pi)^p] = \frac{\Gamma(1 - p\gamma^2/2)}{\Gamma(1 - \gamma^2/2)^p}$$
Consequences

• $M_\gamma(0, 2\pi)$ has density over $\mathbb{R}_+$

\[
\frac{2}{\gamma^2} \Gamma\left(1 - \frac{\gamma^2}{2}\right) x^{-1 - \frac{\gamma^2}{2}} \exp\left(- \Gamma\left(1 - \frac{\gamma^2}{2}\right) x^{-\frac{2}{\gamma^2}}\right) dx
\]

• max du GFF= sum of two independent Gumbel laws

• work done by Guillaume Rémy
Strategy: introduce additional parameters to build an algebraic structure, inducing relations that will solve the problem. This framework is a Conformal Field Theory named Liouville theory.
Plan of the talk

1. Gaussian multiplicative chaos

2. Liouville Conformal Field Theory (LCFT)
Polyakov introduces LCFT:

Polyakov (1981): *Quantum geometry of bosonic strings*.

Conformal Field Theory to solve LCFT:


How do you construct a measure $\mu$ on $\mathbb{R}$? Take a potential $S : \mathbb{R} \to \mathbb{R}$ and define

$$\mu(F) := \int_{\mathbb{R}} F(x) e^{-S(x)} dx$$
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What is a functional integral? It is a measure defined on some functional space $\Sigma$ by

$$\mu(F) = \int_{\Sigma} F(\phi)e^{-S(\phi)} \, D\phi$$

where $D\phi$ is the formal "Lebesgue" measure on the functional space $\Sigma$. 
Quantum Liouville theory on the Riemann sphere

The Riemann sphere can be seen as the complex plane $\mathbb{C}$ equipped with the round metric

$$g(z) = \frac{4}{(1 + |z|^2)^2}.$$

Construct the path integral

$$\int_{\Sigma} F(\phi) e^{-S_L(g,\phi)} D\phi$$

where

$$S_L(\phi) = \frac{1}{4\pi} \int_{\mathbb{C}} (|d\phi|^2_g + 2Q\phi + \mu e^{\gamma\phi}) \, dv_g,$$

$D\phi$ is the formal “Lebesgue” measure on some functional space $\Sigma$ of maps $\phi : \mathbb{C} \to \mathbb{R}$ and

$$\gamma \in (0, 2), \quad \mu > 0 \quad \text{and} \quad Q = \frac{\gamma}{2} + \frac{2}{\gamma}.$$
Correlation functions

Correlation functions of the path integral

\[ \int_{\Sigma} F(\phi) e^{-S_L(\phi)} D\phi \]

are defined for for \( \alpha_i \in \mathbb{R} \) and \( z_i \in \mathbb{C} \)

\[ \langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_{\gamma,\mu} := \int \left( \prod_{i=1}^{n} g(z_i)^{\frac{\alpha_i \Omega}{2}} e^{\alpha_i \phi(z_i)} \right) e^{-S_L(\phi)} D\phi \]
The gradient squared in the potential

\[ S_L(g, \phi) = \frac{1}{4\pi} \int_C \left( |d\phi|^2_g + 2Q\phi + \mu e^{\gamma\phi} \right) dv_g, \]

has a Gaussian interpretation in terms of a log-correlated Gaussian field, called Gaussian Free Field (GFF).

Reformulating

\[ \int F(\phi) \exp \left( - \frac{1}{4\pi} \int_C |d\phi|^2_g dv_g \right) D\phi = C\mathbb{E}[F(X_g)] \]

where \( X \) is a Gaussian field with covariance

\[ \mathbb{E}[X_g(x)X_g(y)] = G(x, y) \]

where \( G \) is the Green function of the Laplacian with zero \( g \)-mean.
Expand any function $\phi$ along eigenfunctions $(e_n)_n$ of Laplacian (with eigenvalue $\lambda_n$)

$$\phi = \sum_n \phi_n e_n$$

In this parametrization $D\phi = \prod_n d\phi_n$ with $d\phi_n$ Lebesgue measure over $\mathbb{R}$.

Then

$$\int_C |d\phi|_g^2 d\nu_g = \sum_n \lambda_n \phi_n^2$$

hence

$$\int F(\phi) \exp \left( - \frac{1}{4\pi} \int_C |d\phi|_g^2 d\nu_g \right) D\phi = \int_{\mathbb{R}^N} F\left( \sum_n \phi_n e_n \right) \prod_n e^{-\frac{\lambda_n \phi_n^2}{2}} d\phi_n$$

$$= C \mathbb{E} \left[ F\left( \sum_n \frac{\alpha_n}{\sqrt{\lambda_n}} e_n \right) \right]$$

with $(\alpha_n)_n$ i.i.d. standard Gaussian.
Path integral definition of LCFT

The existence is based on the following explicit expression:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_{\gamma,\mu} = A \left( \prod_{1 \leq j < k \leq n} \frac{1}{|z_j - z_k|^{\alpha_j \alpha_k}} \right) \mu^{-s} \Gamma(s) \mathbb{E}[Z_1^{-s}]$$

where $$s = \sum_{i=1}^{n} \frac{\alpha_i - 2Q}{\gamma}$$, A some constant (depending on the $$\alpha_i$$ and $$\gamma$$) and

$$Z_1 = \int_{\mathbb{C}} \left( \prod_{i=1}^{n} \frac{1}{|z - z_i|^{\gamma \alpha_i}} \right) g(z)^{1-\frac{\gamma}{4} \sum_{i=1}^{n} \alpha_i} e^{\gamma x_g(z)} (dz)$$

GMC measure
Pick $z_1, \ldots, z_n \in \mathbb{C}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

**Theorem (DKRV, 2014)**

*The correlations exists and are non trivial if and only if:*

$$\forall i, \alpha_i < Q \quad \text{and} \quad Q - \frac{\sum_{i=1}^{n} \alpha_i}{2} < \frac{2}{\gamma} \land \inf_{1 \leq i \leq n} (Q - \alpha_i) \quad (\text{PIC})$$

*In particular, existence implies $n \geq 3!$*
Conformal bootstrap

\[ \langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_{\gamma, \mu} = A \left( \prod_{1 \leq j < k \leq n} \frac{1}{|z_j - z_k|^\alpha_j \alpha_k} \right) \mu^{-s} \Gamma(s) \mathbb{E}[Z_1^{-s}] \]

where \( s = \frac{\sum_{i=1}^{n} \alpha_i - 2Q}{\gamma} \), \( A \) some constant (depending on the \( \alpha_i \) and \( \gamma \)) and

\[ Z_1 = \int_{ \mathbb{C} } \left( \prod_{i=1}^{n} \frac{1}{|z - z_i|^{\gamma \alpha_i}} \right) g(z)^{1 - \frac{\gamma}{4} \sum_{i=1}^{n} \alpha_i} e^{\gamma X_g(z)} (dz) \]

Find an explicit expression for these quantities