α-deformations of an infinite class of continued fraction transformations

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20 September 17
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1. Teaser: Zooming on entropy function
2. Planar natural extensions in the classical cases
3. New set up
4. Main results
5. Synchronization relations
6. Tree of words
7. Group identities and sketch of proofs
8. Lamination relations for 2-D
Thanks to organizers!

Joint with Kari Calta (Vassar College USA) and Cor Kraaikamp (TU Delft, the Netherlands)
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Teaser: Entropies of a collection of interval maps

Entropy, $n = 3$, small alpha

Synchronization for $\alpha$-deformations

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Teaser 2: Entropies, right of a “synchronization interval”

Entropy, \( n = 3 \), right part of first \( r_1 \) branch

Synchronization for \( \alpha \)-deformations
Entropy, \( n = 3 \), right branch, zoom
Regular continued fraction map

\[ x \mapsto \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \]

\([0; a_1, a_2, \ldots] \mapsto [0; a_2, a_3, \ldots].\]
Figure : \((x, y) \mapsto (T(x), \frac{1}{a + y})\) with \(a = \lfloor 1/x \rfloor\).

\[
d\mu = \frac{dx \, dy}{(1 + xy)^2}
\]

Keane ... perhaps Gauss found the invariant measure in a similar way.
For $\alpha \in [0, 1]$, let

$$I_\alpha := [\alpha - 1, \alpha)$$

and

$$T_\alpha(x) := \left| \frac{1}{x} \right| - \left( \left| \frac{1}{x} \right| + 1 - \alpha \right) \quad (x \neq 0),$$

$$T_\alpha(0) := 0.$$
Any $2 \times 2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on reals by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax + b}{cx + d}$.

Let $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For $M \in \text{SL}_2(\mathbb{R})$ and an interval $\mathbb{I}_M$, let

$$T_M(x, y) := \left( M \cdot x, RMR^{-1} \cdot y \right) \text{ for } x \in \mathbb{I}_M, \ y \in \mathbb{R}.$$ 

Thus, $T_M(x, y) = ( M \cdot x, -1/(M \cdot (-1/y)) )$.

The measure $\mu$ on $\mathbb{R}^2$ given by

$$d\mu = \frac{dx \; dy}{(1 + xy)^2}$$

is (locally) $T_M$-invariant.
Any 2 × 2 matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) acts on reals by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax + b}{cx + d} \).

Let \( R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). For \( M \in \text{SL}_2(\mathbb{R}) \) and an interval \( \mathbb{I}_M \), let

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\mathcal{T}_M(x, y) := \begin{pmatrix} M \cdot x, RMR^{-1} \cdot y \end{pmatrix} \quad \text{for} \ x \in \mathbb{I}_M, \ y \in \mathbb{R}.
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Thus, \( \mathcal{T}_M(x, y) = (M \cdot x, -1/(M \cdot (-1/y))) \).

The measure \( \mu \) on \( \mathbb{R}^2 \) given by

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d\mu = \frac{dx \ dy}{(1 + xy)^2}
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is (locally) \( \mathcal{T}_M \)-invariant.
2-D set up

- Any 2 × 2 matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) acts on reals by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax + b}{cx + d} \).

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Each $T_\alpha$ is piecewise Möbius — there is a partition into subintervals, $I_\alpha = \bigcup_\beta K_\beta$, such that $T_\alpha(x) = M_\beta \cdot x$ for all $x \in K_\beta$.

For $x \in K_\beta$ and $y \in \mathbb{R}$ let

$$T_\alpha(x, y) = T_{M_\beta}(x, y) = \left( M_\beta \cdot x, R M_\beta R^{-1} \cdot y \right).$$

We let

$$\Omega_\alpha := \left\{ T^n_\alpha(x, 0) \mid x \in [\alpha - 1, \alpha), \ n \geq 0 \right\}.$$
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Behavior is regulated by the orbits of the endpoints of the interval $\mathbb{I}_\alpha$. 
Initial results about $\alpha$-CF

- Nakada ’81 determined $\Omega_{\alpha}$ for $\alpha \in [1/2, 1]$.
- Kraaikamp ’91 used induction within $\Omega_1$ to confirm Nakada’s work.
- Moussa, Marmi, Cassa ’99 determined $\Omega_{\alpha}$ for $\alpha \in [\sqrt{2} - 1, 1/2)$.

Let $h(T_{\alpha})$ denote the entropy of $T_{\alpha}$, and let $g = (\sqrt{5} - 1)/2$ be the golden mean; with their results, one knew

$$h(T_{\alpha}) = \begin{cases} \frac{\pi^2}{6 \ln(1 + \alpha)} & \text{for } g \leq \alpha \leq 1; \\ \frac{\pi^2}{6 \ln(1 + g)} & \text{for } \sqrt{2} - 1 \leq \alpha \leq g. \end{cases}$$

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Later results

- Let the left and right endpoints of $I_\alpha$ be $l_0(\alpha)$, $r_0(\alpha)$, respectively.

**Theorem (Kraaikamp-S-Steiner '12, Tiozzo et al. '12)**

The set of $\alpha \in (0, 1)$ such that there exists $i = i_\alpha, j = j_\alpha$ with

$$T^i_{\alpha}(r_0(\alpha)) = T^j_{\alpha}(l_0(\alpha))$$

has complement a measure zero Cantor set.

**Theorem (Kraaikamp-S-Steiner '12, Tiozzo et al. '12)**

The $\mu$ measure of the planar models of the natural extensions of $T_\alpha$, and the entropy of these maps, each vary continuously with $\alpha$.

**Theorem (Arnoux-S '13)**

Each $T_\alpha$ is a factor of a return map to a corresponding cross-section for the geodesic flow on the unit tangent bundle of the modular surface.
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Later results

- Let the left and right endpoints of $\mathbb{I}_\alpha$ be $\ell_0(\alpha)$, $r_0(\alpha)$, respectively.

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**Theorem (Arnoux-S '13)**

Each $T_\alpha$ is a factor of a return map to a corresponding cross-section for the geodesic flow on the unit tangent bundle of the modular surface.
Fix $n \geq 3$. Let $\nu = \nu_n = 2 \cos \pi / n$ and $t = 1 + \nu$.

Let $G_n$ be generated by

$$A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \nu & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},$$

(1)

and note that $C = AB$. 

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Fix $\alpha \in [0, 1]$ and define

$$I_\alpha := I_{n, \alpha} = [(\alpha - 1)t, \alpha t).$$

Of endpoints

$$\ell_0 := \ell_0(\alpha) = (\alpha - 1)t$$

and

$$r_0 := r_0(\alpha) = \alpha t$$
Let

$$T_\alpha = T_{n,\alpha} : x \mapsto A^k C^l \cdot x,$$

(2)

- $l > 0$ is minimal such that $C^l \cdot x \notin \mathbb{I}$    Thus, rotate until exit $\mathbb{I}$.
- $k = -\lfloor (C^l \cdot x) / t + 1 - \alpha \rfloor$. Then, translate back into $\mathbb{I}$. 

Let

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(2)

- \( l > 0 \) is \textit{minimal} such that \( C^l \cdot x \notin \mathbb{I} \)  
  Thus, rotate until exit \( \mathbb{I} \).

- \( k = -\lfloor (C^l \cdot x)/t + 1 - \alpha \rfloor \).  
  Then, translate back into \( \mathbb{I} \).
\[
x = \frac{1}{1 - \frac{1}{1 + kt - T_\alpha(x)}}.
\]
Theorem 1

For $n \geq 3$, the set of $\alpha \in (0,1)$ such that there exists $i = i_\alpha, j = j_\alpha$ with

$$T_{n,\alpha}^i(r_0(\alpha)) = T_{n,\alpha}^j(\ell_0(\alpha))$$

is of full Lebesgue measure.

Call the set of these $\alpha$ the synchronization set for $n$. 
Theorem 1, more precision

For \( n \geq 3 \), the synchronization set is the union of intervals, 
\[ J_{k,v} = [\zeta_{k,v}, \eta_{k,v}] \] with \( k \in \mathbb{Z} \setminus \{0\} \) and \( v \in \mathcal{V} \), a tree of words defined below. The complement of the union of the \( [\zeta_{k,v}, \eta_{k,v}] \) is a measure zero Cantor set.
For $k \in \mathbb{N}$ and $v \in \mathcal{V}$ (defined below), let $\mathcal{I}_{k,v} = [\zeta_{k,v}, \eta_{k,v}]$.

**Theorem**

Fix $n \geq 3$, $k \in \mathbb{N}$, $v \in \mathcal{V}$ and $\alpha \in (\zeta_{k,v}, \eta_{k,v})$.

There is a connected union of finitely many rectangles $\Omega_{n,\alpha}$ upon which $T_{n,\alpha}$ is bijective, up to $\mu$-measure zero.

Furthermore, this gives the natural extension of $T_{n,\alpha}$.

Moreover, the collection of heights (top and bottoms) of the rectangles comprising $\Omega_{n,\alpha}$ depends only on $(n, k, v)$.

Off of synchronization set, each $\Omega_{\alpha}$ given by union of infinitely many rectangles, have continuity in that vertices converge.
For \( k \in \mathbb{N} \) and \( v \in \mathcal{V} \) (defined below), let \( J_{k,v} = [\zeta_{k,v}, \eta_{k,v}] \).

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Theorem 2

For \( k \in \mathbb{N} \) and \( v \in \mathcal{V} \) (defined below), let \( \mathcal{I}_{k,v} = [\zeta_{k,v}, \eta_{k,v}] \).

**Theorem**

Fix \( n \geq 3, k \in \mathbb{N}, v \in \mathcal{V} \) and \( \alpha \in (\zeta_{k,v}, \eta_{k,v}) \).

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Off of synchronization set, each \( \Omega_{\alpha} \) given by union of infinitely many rectangles, have continuity in that vertices converge.
Figure: The domain $\Omega_{3,0,14}$, with blocks $B_i$ (projecting to cylinders for $T_\alpha$), and their images, both denoted by $i$. Here $R_{k,v} = AC$ and $L_{k,v} = A^{-1}CA^{-2}CA^{-2}CA^{-1}CA^{-1}$, and $\alpha$ is an interior point of $J_{1,1}$. 

One $\Omega_{n,\alpha}$
Cylinders

\[
(\ell_0, -1, 1, k, 1, \ldots, r_0, b, 1, n_0, \ldots)
\]

**Figure**: Schematic representation of cylinders for three values of \(\alpha\) (here \(n = 3\)).
Fix $n$. Let $S = S_n$ be given as

$$S : \bigcup_{\alpha \in [0,1]} \{r_0(\alpha)\} \times \mathbb{I}_\alpha \to \bigcup_{\alpha \in [0,1]} \{r_0(\alpha)\} \times \mathbb{I}_\alpha$$

$$(r_0(\alpha), y) \mapsto (r_0(\alpha), T_\alpha y)$$

Recall that $r_0(\alpha) = \alpha t$ and $\ell_0(\alpha) = (\alpha - 1)t = r_0(\alpha) - t$. 
Fix $n$. Let $S = S_n$ be given as

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Recall that $r_0(\alpha) = \alpha t$ and $\ell_0(\alpha) = (\alpha - 1)t = r_0(\alpha) - t$. 
Figure: The unions of the various cylinders for the $T_{n,\alpha}$ form cylinders for $S_n$. Each $\mathbb{I}_\alpha$ is given as a vertical fiber, with its left endpoint $\ell_0(\alpha) = \alpha t - t$ at the bottom and its right endpoint $r_0(\alpha) = \alpha t$ at the top. Here: $n = 3$. 
Figure: The graph of $x \mapsto T_{3,\alpha}(x - t)$, with $x = \alpha t$, thus the values of $\ell_1(\alpha)$. In red that of $x \mapsto T_{3,\alpha}(x)$; the red curves give $r_1(\alpha)$. (Here $t = t_3 = 2$.) Gray vertical lines demarcate natural partition; to left of leftmost gray vertical line “$C^2$ never appears.”
Figure: Zoom in on first red branch in the “no $C^2$” zone. Red gives the single branch of $y = r_1(\alpha)$ while blue colors the two branches of $y = \ell_4(\alpha)$ for $x$-range plotted. The $x$-axis is shown as a dotted line.
In the previous figure, find

\[ r_1 = C^{-1}A^{-1}C \cdot \ell_4 \]

holds for \( \alpha \in [\zeta, \eta) \).

For some \( u \),

\[ r_2 = A^u C \cdot r_1 = A^{u-1} C \cdot \ell_4 = \ell_5. \]

This must give \( \ell_5 \) because we are in region of “no \( C^2 \)” and there is a unique translation of \( C \cdot \ell_4 \) into \( \mathbb{I}_\alpha \).
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A synchronization relation

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This must give \(\ell_5\) because we are in region of “no \(C^2\)” and there is a unique translation of \(C \cdot \ell_4\) into \(\mathbb{I}_\alpha\).
Moral: synchronization occurs on intervals for which there are group elements $R, L$ are such that

- $R, L$ satisfy a synchronization relation and

- $r_{j-1} = R \cdot r_0(\alpha)$ and $\ell_{i-1} = L \cdot \ell_0(\alpha)$
Moral: synchronization occurs on intervals for which there are group elements $R, L$ are such that

- $R, L$ satisfy a synchronization relation and
- $r_{j-1} = R \cdot r_0(\alpha)$ and $\ell_{i-1} = L \cdot \ell_0(\alpha)$
Relation reveals further digits of $r_0$ at right endpoint

At the right endpoint of $J_{k,v}$, relation gives

$$r_{j-1} = C^{-1}A^{-1}C \cdot r_0 \quad \text{or} \quad C^{-1}AC \cdot r_{j-1} = r_0$$

Since $r_1 = A^kC \cdot r_0$,

$$r_j = r_1$$

and

$$r_j = A^{k+1}C \cdot r_{j-1}$$
Relation reveals further digits of $r_0$ at right endpoint

At the right endpoint of $\mathcal{J}_{k,v}$, relation gives

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Period of $r_0(\eta_{k,v})$ — only $k, k + 1$ as digits

- In parameter region where “$C^2$ does not appear”. Use simplified digits, for $\alpha \in \mathcal{J}_{k,v}$,

$$r_0(\alpha) = \underbrace{k^{c_1}, (k + 1)^{d_1}, \ldots, (k + 1)^{d_{s-1}}, k^{c_s}}_{d(k,v), \; v = c_1 d_1 \cdots c_{s-1} d_{s-1} c_s}, \ldots$$

- At right endpoint $\eta_{k,v}$ find periodic

$$r_0(\alpha) = d(k, v), k + 1, k^{c_1-1}, (k + 1)^{d_1}, \ldots, (k + 1)^{d_{s-1}}, k^{c_s}$$
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For each $s > 1$ and each word $\nu = c_1 d_1 \cdots c_{s-1} d_{s-1} c_s$, define

$$\nu' = \begin{cases} 
1(c_1 - 1)d_1 c_2 \cdots c_{s-1} d_{s-1} c_s & \text{if } c_1 \neq 1, \\
(d_1 + 1)c_2 \cdots c_{s-1} d_{s-1} c_s & \text{otherwise}.
\end{cases}$$

(When $\nu = c$ with $c > 1$ then let $\nu' = 1(c - 1)$, and when $\nu = 1$ then $\nu' = 1$.)
Operators $\Theta_q$

Set

\[ \Theta_{-1}(c_1) = c_1 + 1 \]
\[ \Theta_q(1) = 1q1 \text{ for } q \geq 1 \]
\[ \text{For } c > 1, \text{ set } \Theta_q(c) = c[1(c - 1)]^q1c \text{ for any } q \geq 0. \]

Recursively ... Suppose \( v = \Theta_p(u) = uv'' \) for some \( p \geq 0 \) and some suffix \( v'' \). Then define for any \( q \geq 0 \)

\[ \Theta_q(v) = v(v')^q v''. \]

This is a palindrome; it is shortest “self-dominant” word extending \( v(v')^q \) which is larger than \( v(v')^\infty \).

Let \( \mathcal{V} \) be the tree of all words obtained starting from \( v = 1 \).
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Figure: Each vertex of the directed tree $\mathcal{V}$ has countably infinite valency. A small portion of $\mathcal{V}$ with a hint of the derived words map, $\mathcal{D}$.
For $k \in \mathbb{N}, \nu \in \mathcal{V}$, let

$$I_{k,\nu} = \{ \alpha \mid r_0(\alpha) \text{ has digits } d(k, \nu) \}$$

This is partitioned

$$I_{k,\nu} = I_{k,\nu} \cup \bigcup_{q=q'}^\infty I_{k,\Theta_q(\nu)},$$

where $q' = 0$ unless $\nu = c_1$, in which case $q' = -1$. 

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Synchronization for $\alpha$-deformations
Partitioning with the $J_{k,v}$

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Figure: A hint of the partition of the interval where $r_0(\alpha) = d(k,v) \cdots$, denoted here $I_{k,v}$. 

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Synchronization for $\alpha$-deformations
Right endpoint of $\alpha$-cylinder $\mathcal{I}_{k,v}$

Figure: A non-full branch. Here $n = 3, v = 111$ and $k = 1$; we have that $\omega_{1,111}$ is determined by the fixed point of $R_{1,11}$. The labels $L, R$ mark respectively the curves $y = L_{1,111} \cdot r_0(\alpha), y = R_{1,111} \cdot r_0(\alpha)$ where $\alpha = x/2 = x/t_{3,3}$. Red gives of $y = r_3(\alpha)$, while blue gives $y = \ell_9(\alpha)$; Magenta gives the branches of $y = r_2(\alpha)$. The left portion has $0.3582 < x < 0.3592$. The right “zooms in” to $0.35910 < x < 0.35915$. (This interval lies between the vertical gray lines in both portions.)

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Order on cylinders is $k > k + 1$, gives order on (shifts of) words: any $c_j$ greater than any $d_i$, usual order of integers for $c_j$, reverse for $d_i$

Define full branch prefix $f(v)$ as longest prefix $u$ of $v$ such that $u^\infty$ is maximal among all prefixes.

Find right endpoint of $\mathcal{I}_{k,v}$ has $r_0(\alpha)$ of digits $d(k,f(v))^\infty$.

One shows

$$f(\Theta_q(v)) = (\Theta_{q-1}(v))'.$$

Can then prove partition result.
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Synchronization relation implies $\ell_0$ digits $-1, -2$

Let $W = A^{-2}C (A^{-1}C)^{n-3} A^{-2}C(A^{-1}C)^{n-2}$.

**Lemma (one step)**

For $c, k \geq 1$,

$$(A^kC)^c = C^{-1}A^{-1}C (A^{-1}C)^{n-2} [W^{k-1}A^{-2}C(A^{-1}C)^{n-3}]^{c-1} W^k A^{-1}.$$  

**Lemma (glueing)**

For $k \geq 1$,

$$WA^{-1} \cdot A^k CA^{-1}C = A^{-2}C (A^{-1}C)^{n-3} W^k A^{-2}C.$$
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**synchr. rel.**

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Outline of proof of Synchronization on full measure of $\alpha$.

- Partition result holds, due to descriptions of $\zeta_{k,v}, \eta_{k,v}, \omega_{k,v}$
- Lemmas on previous slide give necessary $\ell_0$ digits for synchronization on $\mathcal{J}_{k,v}$.
- Induction shows admissibility of these $\ell_0$ digits. Of course, not admissible to right, but relation helps.
- Since only $-1, -2$ can use $\alpha = 0$ maps (actually with acceleration for finite measure from Calta-S), get complement of measure zero.
- Easily show no other $\alpha$ have synchronization (thus exact description of the complement ... follow branch of tree).
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Outline of proof of Synchronization on full measure of $\alpha$.

- Partition result holds, due to descriptions of $\zeta_{k,v}, \eta_{k,v}, \omega_{k,v}$.
- Lemmas on previous slide give necessary $l_0$ digits for synchronization on $J_{k,v}$.
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- Since only $-1, -2$ can use $\alpha = 0$ maps (actually with acceleration for finite measure from Calta-S), get complement of measure zero.
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One $\Omega_{n,\alpha}$, again

Figure: The domain $\Omega_{3,0.14}$, with blocks $B_i$ (projecting to cylinders for $T_\alpha$), and their images, both denoted by $i$. Here $R_{k,v} = AC$ and $L_{k,v} = A^{-1}CA^{-2}CA^{-2}CA^{-1}CA^{-1}$, and $\alpha$ is an interior point of $\mathcal{I}_{1,1}$. 
A second $\Omega_{k,v}$

Figure: The domain $\Omega_{3,0.86}$. Blocks $B_{i,j}$ and their images, both denoted by $(i,j)$. Here $L_{-k,v} = A^{-2}CA^{-1}$ and $R_{-k,v} = ACAC^2$, and $\alpha$ is an interior point of $\mathcal{J}_{-2,1}$. Also, hints as to the lamination ordering.
Connectedness of $\Omega_{k,v}$ requires relations on heights

\[ y_1 = y_{\tau(0)} \xrightarrow{-k} y_{\tau(1)} \xrightarrow{-k} \cdots \xrightarrow{-k} y_{\tau(\iota)} = y_S \xrightarrow{-k-1} \cdots \xrightarrow{-k} y_{\tau(S-1)} \xrightarrow{-k} y_{\tau(S)} = y_{S+1} \]

\[ y_{\beta(3)} \leftarrow y_{\beta(3-1)} \leftarrow \cdots \leftarrow y_{\beta(1)} \leftarrow y_{\beta(0)} = y_{-1} \]

Figure: Relations on the heights of rectangles for general $-k$, $v$ and $\alpha \in (\eta_{-k,v}, \delta_{-k,v})$. The red paths are used to prove that lamination occurs. Horizontal arrows used to show that boundaries are sent to boundaries.
THANK YOU!