Mesure harmonique et multifractalité du SLE (I)

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Analyse Multifractale
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Integral Means Spectrum

- Let $\Omega$ be a bounded simply connected domain containing 0 and $K = \partial \Omega$; then the harmonic measure from 0, $\omega^0$, is the image of normalized Lebesgue measure on the unit circle by the Riemann map $\Phi$ from the unit disk onto $\Omega$ such that $f(0) = 0$. We have a similar description if $\Omega = \mathbb{C} \setminus K$ where $K$ is a compact connected set containing at least two points: in this case $\omega^\infty$ is the image of normalized Lebesgue measure on the unit circle by a Riemann map from $\mathbb{C} \setminus \mathbb{D}$ onto $\Omega$ fixing $\infty$.

- The integral means of $\Phi$ are
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  \mathcal{I}(r, p, \Phi) := \int_0^{2\pi} |\Phi'(re^{i\theta})|^p \, d\theta, \quad 0 < r < 1, \, (r > 1) \, p \in \mathbb{R};
  \]

- Average:
  \[
  \mathbb{E} \mathcal{I}(r, p, \Phi) := \int_0^{2\pi} \mathbb{E} \left[ |\Phi'(re^{i\theta})|^p \right] \, d\theta, \quad 0 < r < 1, \, (r > 1).
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- One then defines
  \[
  \beta_\Phi(p) := \limsup_{r \to 1^\pm} \frac{\log(\mathcal{I}(r, p, \Phi))}{\log(\frac{1}{|1-r|})};
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-One passes from $\beta$ to $f$ by a Legendre transform,

$$\frac{1}{\alpha} f(\alpha) = \inf_p \left\{ \beta(p) - p + 1 + \frac{1}{\alpha} p \right\},$$

$$\beta(p) = \sup_\alpha \left\{ \frac{1}{\alpha} (f(\alpha) - p) \right\} + p - 1.$$
Universal Integral Means Spectrum

- $B(p) = \sup\{\beta_\Phi(p), \Phi \in S\}$.
- $B_{\text{bd}}(p) = \sup\{\beta_\Phi(p), \Phi \in S, \Phi \text{ bounded}\}$.
- Theorem (Makarov):

$$B(p) = \max\{B_{\text{bd}}(p), 3p - 1\}.$$
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We wish to have a unified treatment of the bounded and the unbounded cases.

Idea: $\Phi \in \mathcal{S} \Rightarrow \Psi(z) = \frac{1}{\phi}(1/z)$ is bounded (near the unit circle),

$$|\psi'(z)|^p = \frac{|\phi'(1/z)|^p}{|z^2\phi(1/z)|^{2p}}.$$
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- If the limit exists,

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As we shall see, one can read various standard spectra in the \((p, q)\) plane:

- The usual integral means spectrum on the line \(q = 0\),
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![Graph showing the spectrum in the $(p, q)$ plane.](image-url)
As we shall see, one can read various standard spectra in the \((p, q)\) plane:

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Universal Generalized Integral Means Spectrum

- We can similarly define a \textit{universal generalized} integral means spectrum.

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B(p, q) = \max \{B_{bd}(p), 3p - 2q - 1\}.
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Radial Loewner Evolution

Let $\Delta$ be the complement of the closed unit disk and $\gamma : [0, +\infty[ \rightarrow \mathbb{C}$ be an injective and continuous function such that $\gamma(0) \in \partial \Delta$ and $\gamma([0, +\infty[) \subset \Delta$. We define

$$\Omega_t = \Delta \setminus \gamma([0, t])$$

and $f_t$ the Riemann map from $\Delta$ onto $\Omega_t$ such that $f_t(\infty) = \infty$ and $f_t'(\infty) > 0$. We normalize the process by assuming $f_0'(\infty) = 1$ and then, changing time if necessary, we assume $f_t'(\infty) = e^t$.

We define also $g_t = f_t^{-1}$ and $\lambda(t) = g_t(\gamma(t))$.

We have the semi-group type identity

$$f_t = f_s \circ h_{s,t}$$
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1) Key fact: $f_t$ is locally Lipschitz wrt $t$.

2) Using the semi-group identity, Schwarz reflection theorem and Cauchy formula we find that $f_t$ obeys the following PDE

$$\frac{\partial f_t}{\partial t}(z) = tf_t'(z) \frac{z + \lambda(t)}{z - \lambda(t)}.$$
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- Converse: the function $h_{s,t}(z) = f_s^{-1} \circ f_t(z)$, as a function of $s$, is a solution of the differential equation

$$\frac{dw}{ds} = -w \frac{\lambda(s) + w}{\lambda(s) - w}, \ w(t) = z. \quad (1)$$

- Since

$$\frac{d|w|^2}{ds} = -2|w|^2 \Re \frac{\lambda(s) + w}{\lambda(s) - w}$$

the modulus of a solution is decreasing, which implies that this equation has a solution $s \mapsto w(s; t, z)$ defined on $[0,t]$.

- By Cauchy-Lipschitz theorem, this function is injective in $z$ and $f_t = \varphi(w(0; t, z))$ is the solution to Loewner equation such that $f_0 = \varphi$. 
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Radial Loewner Evolution
Whole-plane SLE

\[ \lambda(t) \]

\[ z_t = f_t^{-1}(\infty) \]

\[ \gamma(t) = f_t(\lambda(t)) \]

\[ \gamma(0) = f_0(1) \]

\[ f_t(0) = 0 \]
Chordal SLE

\[ w = f_t(z) \]

\[ f_t(x') \rightarrow f_t(x) \]

\[ x' \quad 0 \quad x \]

\[ 0 \quad \eta_t \]
Phases for SLE
Physical interpretation
The Brownian Frontier
Integrable red parabola

Let $f$ be a time 0 whole-plane (inner) $\text{SLE}_\kappa$, and $(p, q) \in \mathbb{R}^2$.

$$F(z) := \mathbb{E} \left( f'(z)^p \left( \frac{Z}{f(z)} \right)^{\frac{q}{2}} \right), \quad G(z, \bar{z}) := \mathbb{E} \left( |f'(z)|^p \left| \frac{Z}{f(z)} \right|^q \right).$$

The red parabola has for parameterization,

$$p(\gamma) = (2 + \frac{\kappa}{2})\gamma - \frac{\kappa}{2}\gamma^2, \quad \gamma \in \mathbb{R},$$

$$q(\gamma) = (3 + \frac{\kappa}{2})\gamma - \kappa\gamma^2.$$

Theorem: If $p = p(\gamma)$ and $q = q(\gamma)$, one has

$$F(z) = (1 - z)^\gamma, \quad G(z, \bar{z}) = \frac{(1 - z)^\gamma(1 - \bar{z})^\gamma}{(1 - z\bar{z})^{\frac{\kappa\gamma^2}{2}}}.$$
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$$F(z) = (1 - z)^\gamma, \quad G(z, \bar{z}) = \frac{(1 - z)^\gamma(1 - \bar{z})^\gamma}{(1 - z\bar{z})^{\kappa\frac{\gamma^2}{2}}}.$$
The starting point is to consider the radial $SLE_\kappa$, that is the solution to the ODE

$$\partial_t g_t(z) = g_t(z) \frac{\lambda(t) + g_t(z)}{\lambda(t) - g_t(z)}, \; z \in \mathbb{D},$$

with the initial condition $g_0(z) = z$, and where $\lambda(t) = e^{i\sqrt{\kappa} B_t}$.

The (conjugate, inverse) radial SLE process $\tilde{f}_t$ is defined for $t \geq 0$ as

$$\tilde{f}_t(z) := g_t^{-1}(z \lambda(t))/\lambda(t). \quad (2)$$
Idea of proof (for $p = q = 1$)

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$\tilde{f}_s(z) = \hat{g}_{-s}(z),$

where $\hat{g}_t(z) = g_{s+t}(g_s^{-1}(z\lambda(s))))/\lambda(s)$ is shown to be a radial SLE.

$\tilde{f}_t(z) = \lambda(s)\tilde{f}_{t-s}(\tilde{f}_s(z)/\lambda(s)).$

The limit in law, $\lim_{t \to +\infty} e^t \tilde{f}_t(z)$, exists, and has the same law as the (time zero) interior whole-plane random map $f_0(z)$.

$F(z) := \mathbb{E} \left( \frac{f'(z)}{f(z)} \right), \quad (3)$

$\tilde{F}(z, t) := \mathbb{E} \left( \frac{\tilde{f}'_t(z)}{\tilde{f}_t(z)} \right). \quad (4)$

$\lim_{t \to \infty} \tilde{F}(z, t) = F(z).$
Idea of proof (for $p = q = 1$)

\[ \tilde{f}_s(z) = \hat{g}_{-s}(z), \]

where $\hat{g}_t(z) = g_{s+t}(g_s^{-1}(z\lambda(s))))/\lambda(s)$ is shown to be a radial SLE.

\[ \tilde{f}_t(z) = \lambda(s)\tilde{f}_{t-s}(\tilde{f}_s(z)/\lambda(s)). \]

The limit in law, $\lim_{t \to +\infty} e^t \tilde{f}_t(z)$, exists, and has the same law as the (time zero) interior whole-plane random map $f_0(z)$.

\[ F(z) := \mathbb{E} \left( z \frac{f'(z)}{f(z)} \right), \quad (3) \]

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$\lim_{t \to \infty} \tilde{F}(z, t) = F(z)$. 
Idea of proof \((p = q = 1)\)

\(\Rightarrow\) For \(s \leq t\), define \(\mathcal{M}_s := \mathbb{E}\left(\frac{\tilde{f}'_t(z)}{\tilde{f}_t(z)} | \mathcal{F}_s\right)\), where \(\mathcal{F}_s\) is the \(\sigma\)-algebra generated by \(\{B_u, u \leq s\}\). \((\mathcal{M}_s)_{s \geq 0}\) is by construction a martingale. Because of the Markov property of SLE, we have

\[
\mathcal{M}_s = \mathbb{E}\left(\frac{\tilde{f}'_t(z)}{\tilde{f}_t(z)} | \mathcal{F}_s\right) = \mathbb{E}\left(\frac{\tilde{f}'_s(z)}{\lambda(s)} \frac{\tilde{f}'_{t-s}(\tilde{f}_s(z)/\lambda(s))}{\tilde{f}'_{t-s}(\tilde{f}_s(z)/\lambda(s))} | \mathcal{F}_s\right)
\]

\[
= \frac{\tilde{f}'_s(z)}{\lambda(s)} \mathbb{E}\left(\frac{\tilde{f}'_{t-s}(\tilde{f}_s(z)/\lambda(s))}{\tilde{f}'_{t-s}(\tilde{f}_s(z)/\lambda(s))} | \mathcal{F}_s\right)
\]

\[
= \frac{\tilde{f}'_s(z)}{\tilde{f}_s(z)} \tilde{F}(z_s, \tau),
\]

\(\Rightarrow\) where \(z_s := \tilde{f}_s(z)/\lambda(s)\), and \(\tau := t - s\).
Idea of proof \((p = q = 1)\)

- For \(s \leq t\), define \(\mathcal{M}_s := \mathbb{E}\left(\frac{\tilde{f}'_t(z)}{\tilde{f}_t(z)} \mid \mathcal{F}_s\right)\), where \(\mathcal{F}_s\) is the \(\sigma\)-algebra generated by \(\{B_u, \ u \leq s\}\). \((\mathcal{M}_s)_{s \geq 0}\) is by construction a martingale. Because of the Markov property of SLE, we have

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\]

\[
= \frac{\tilde{f}'_s(z)}{\lambda(s)} \mathbb{E}\left(\frac{\tilde{f}'_{t-s}(\tilde{f}_s(z)/\lambda(s))}{\tilde{f}'_{t-s}(\tilde{f}_s(z)/\lambda(s))} \mid \mathcal{F}_s\right)
\]

\[
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\]

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Idea of proof ($p = q = 1$)

- For $s \leq t$, define $\mathcal{M}_s := \mathbb{E}\left( \frac{\tilde{f}'_t(z)}{\tilde{f}_t(z)} | \mathcal{F}_s \right)$, where $\mathcal{F}_s$ is the $\sigma$-algebra generated by $\{B_u, \ u \leq s\}$. $(\mathcal{M}_s)_{s \geq 0}$ is by construction a martingale. Because of the Markov property of SLE, we have

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\]

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Idea of proof \( (p = q = 1) \)

\[
\partial_s \log \tilde{f}'_s = \frac{\partial_z \left[ \tilde{f}_s \frac{\tilde{f}_s + \lambda(s)}{\tilde{f}_s - \lambda(s)} \right]}{\tilde{f}'_s} = \frac{\tilde{f}_s + \lambda(s)}{\tilde{f}_s - \lambda(s)} - \frac{2\lambda(s)\tilde{f}_s}{(\tilde{f}_s - \lambda(s))^2} \tag{5}
\]

\[= 1 - \frac{2}{(1 - z_s)^2},\]

\[
\partial_s \log \tilde{f}_s = \frac{\partial_s \tilde{f}}{\tilde{f}_s} = \frac{z_s + 1}{z_s - 1}, \tag{6}
\]

\[
dz_s = z_s \left[ \frac{z_s + 1}{z_s - 1} - \frac{\kappa}{2} \right] ds - iz_s \sqrt{\kappa} dB_s. \tag{7}
\]

The coefficient of the \( ds \)-drift term of the Itô derivative of \( M_s \) is obtained from the above as,

\[
\tilde{f}'_s(z) \left[ -\frac{2z_s}{(1 - z_s)^2} + z_s \left( \frac{z_s + 1}{z_s - 1} - \frac{\kappa}{2} \right) \partial_z - \partial_\tau - \frac{\kappa}{2} z_s^2 \partial_z^2 \right] \tilde{G}(z_s, \tau), \tag{8}
\]
Idea of proof \((p = q = 1)\)
\[
\begin{align*}
\partial_s \log \tilde{f}'_s &= \frac{\partial_z \left[ \tilde{f}_s \frac{\tilde{f}_s + \lambda(s)}{\tilde{f}_s - \lambda(s)} \right]}{\tilde{f}'_s} = \frac{\tilde{f}_s + \lambda(s)}{\tilde{f}_s - \lambda(s)} - \frac{2\lambda(s)\tilde{f}_s}{(\tilde{f}_s - \lambda(s))^2} \quad (5) \\
&= 1 - \frac{2}{(1 - z_s)^2},
\end{align*}
\]
\[
\begin{align*}
\partial_s \log \tilde{f}_s &= \frac{\partial_s \tilde{f}_s}{\tilde{f}_s} = \frac{z_s + 1}{z_s - 1}, \\
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\[
\begin{align*}
\frac{\tilde{f}'_s(z)}{\tilde{f}_s(z)} \left[ -\frac{2z_s}{(1 - z_s)^2} + z_s \left( \frac{z_s + 1}{z_s - 1} - \frac{\kappa}{2} \right) \partial_z - \partial_\tau - \frac{\kappa}{2} z_s^2 \partial^2_z \right] G(z_s, \tau), 
\end{align*}
\]
Idea of proof (for $p = q = 1$)

\[
-\frac{2z}{(1 - z)^2} + z \left( \frac{z + 1}{z - 1} \right) \partial_z - \frac{\kappa}{2} (z \partial_z)^2 \right] (F)(z) = 0. \tag{9}
\]

In the general case we obtain with the same method:

\[
\left[ -\frac{\kappa}{2} (z \partial_z)^2 - \frac{1 + z}{1 - z} z \partial_z - \frac{p}{(1 - z)^2} + \frac{q}{1 - z} + p - q \right] (F)(z) = 0. \tag{10}
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Idea of proof (for $p = q = 1$)

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\]
The red parabola: idea of proof

Using Itô calculus, we derive a partial differential equation satisfied by $G$,

$$
\mathcal{P}(D)[G(z, \bar{z})] = \left[ -\frac{\kappa}{2}(z \partial_z - \bar{z} \partial_{\bar{z}})^2 - \frac{1 + z}{1 - z} z \partial_z - \frac{1 + \bar{z}}{1 - \bar{z}} \bar{z} \partial_{\bar{z}} - \frac{p}{(1 - z)^2} - \frac{p}{(1 - \bar{z})^2} + \frac{q}{1 - z} + \frac{q}{1 - \bar{z}} + 2(p - q) \right] G(z, \bar{z}) = 0.
$$
The red parabola: idea of proof

- Set $\varphi_\gamma(z) := (1 - z)^\gamma$:
  - $\mathcal{P}(\partial_z)[\varphi_\gamma] = A(p, q, \gamma)\varphi_\gamma + B(q, \gamma)\varphi_{\gamma-1} + C(p, \gamma)\varphi_{\gamma-2}$, with $A$, $B$ and $C$ involving monomials in $p$, $q$, in addition to quadratic polynomials in $\gamma$.
  - $A + B + C \equiv 0$.
  - The equations, $A = C = 0$, exactly correspond to $(p, q)$ on the red parabola, and directly yield its parameterization in terms of $\gamma$.
  - If $F = \varphi_\gamma$, then we seek $G$ as
    \[
    G(z, \bar{z}) = \varphi_\gamma(z)\varphi_\gamma(z) P(z\bar{z}),
    \]
    and writing $\mathcal{P}(D)[G] = 0$ leads to a simple differential equation for $P$. 
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When $q = 0$ and $p = 2$ Parseval identity allows us to rephrase the latter result as identities for the coefficients $a_n$ where we have written

$$f(z) = z + \sum_{n \geq 2} a_n z^n.$$

$p = 2$, $p = 2$ occurs for $\kappa = 2, 6$ with $\gamma = 2, 1$ respectively and we obtain the remarkable identities:

- $\kappa = 2$ : $\mathbb{E}(|a_n|^2) = n$.
- $\kappa = 6$ : $\mathbb{E}(|a_n|^2) = 1$. 
Coefficient problem

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